THE UNIVERSITY OF HONG KONG

COMP4801 Interim Report

Fixed-Price Mechanisms in Bilateral Trade

Research on Algorithm Design and Analysis

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Abstract

This project studies social welfare maximization in bilateral trade. The model comprises one seller, one buyer, and one item they want to trade. As research shows that no truthful mechanisms can be socially efficient, there has been extensive work on approximating optimal social welfare. In this project, our work focuses on two recent studies that give the best-known approximation ratio using Fixed-Price Mechanisms. The goal is to follow their analysis and use more efficient programs on this problem. In the second part of this project, we also work on the multi-unit bilateral trad, where the seller holds multiple identical items.

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Chapter 1

Introduction

The bilateral trade model studies that one buyer and one seller trade an indivisible item. Research shows that no incentive-compatible, individually rational, and budget-balanced mechanism can achieve the optimal social welfare [6]. Motivated by this impossibility result, many researchers have been working on approximately efficient mechanisms. The Fixed-Price Mechanism posts a price p, and trade occurs if both agents accept this price. Two recent studies respectively show that there exists a fixed price mechanism that guarantees 0.71 [5] and 0.72 [3] fraction of the optimal social welfare for any buyer and seller value distributions, and no fixed price mechanism can achieve more than 0.7381 fraction of the optimal welfare. In the first phase of the project, our work is to do an in-depth analysis of their methods, follow their analysis and try to improve the efficiency and preciseness of the approaches. In the second phase of the project, we will work on the generalized bilateral trade setting with multiple identical items.

1.1 Background

This section provides relevant background information on the Bilateral Trade Model, Fixed-Price Mechanisms, approximation algorithms, and the Multi-unit Bilateral Trade Model.

Bilateral Trade Model In the bilateral trade model, a seller owns an indivisible item, and a buyer wants to purchase it. Both agents have private valuations about the item, we denote the seller's valuation as s and the buyer's valuation as b. We only know the public probability distribution (cumulative distribution functions) from which the valuations are drawn, i.e., $s \sim F_S$, $b \sim F_B$. A socially efficient mechanism always trades when the buyer has higher values, and never trades if the seller has higher values.

Fixed-Price Mechanisms The Fixed-Price Mechanism posts p as the price. Individual Rationality requires that $s \leq p \leq b$, that is, both agents accept the price if and only if they expect non-negative payoffs. In some cases, agents may lie about their valuations in order to gain more benefits from trade. The Fixed-Price Mechanism is Dominant Strategy Incentive Compatible, agents have no incentive to deviate from using the strategy of their true valuations. This mechanism is also Strongly Budget Balanced, that is, the payment of the buyer is fully transferred to the seller.

Social Welfare and Approximation Ratio The social welfare is b if the trade occurs and s otherwise. The optimal social welfare is defined as $\mathrm{OPT} = \mathbb{E}_{s\sim F_S, b\sim F_B}[b + (b - s) \cdot 1_{s\leq b}]$. The welfare of the Fixed-Price Mechanism using p as the price is $\mathrm{ALG} = \mathbb{E}_{s\sim F_S, b\sim F_B}[b + (b - s) \cdot 1_{s\leq p\leq b}]$.

The approximation ratio describes the performance of the mechanism on the worst-case instance:

$$\alpha = \min_{I=(F_S,F_B)} \frac{\mathrm{ALG}(I)}{\mathrm{OPT}(I)}$$

To maximize the above ratio is equivalent to maximize the welfare obtained by the mechanism.

Multi-unit Bilateral Trade Model The Multi-unit Bilateral Trade Model studies one buyer and one seller who holds k identical items. Both agents have increasing submodular valuations for the number of units they own. An instance is a distribution over seller and buyer valuation functions. The mechanism M

needs to decide the number of items q transferred from the seller to the buyer and the payment p transferred from the buyer to the seller.

1.2 Related Research

In 1983, Myerson and Satterthwaite [6] initiated the study on the bilateral trade model. Their work shows that as long as the buyer and seller distributions are continuous with positive probability densities and the value distribution intervals overlap, no truthful mechanism can be socially efficient. Many researchers have been working on exploring what is possible.

Some fixed price mechanisms are efficient in approximating social welfare. The Median Mechanism [2] sets the median of the seller distribution as the price. This mechanism provides a $\frac{1}{2}$ approximation. The Random Quantile Mechanism [1] chooses a quantile x randomly from a certain distribution and outputs the x-quantile of the seller distribution as the price. This mechanism is an $1 - \frac{1}{e}$ approximation. The Optimal Fixed Price Mechanism [5] [3] posts a price that maximizes the welfare of the mechanism. The approximation ratio is around 0.71 and 0.72 respectively. In terms of using fixed price mechanisms to approximate welfare, this mechanism is the most efficient.

[4] is on the Multi-unit Bilateral Trade Model. They show Generalized Median Mechanism provides a $\frac{1}{2}$ approximation and Generalized Random Quantile Mechanism is a $1 - \frac{1}{e}$ approximation in this setting.

1.3 Motivation and Objectives

Work Related to Fixed-Price Mechanisms Two papers [5] [3] on Fixed-Price Mechanisms that give the best known bound of 0.71 and 0.72 both apply the method of numerically solving a program to approach the approximation ratio, leaving a relatively small gap from the impossibility result of 0.7381. They also state in the paper that the lower bound and the upper bound can converge with more computational resources. In this project, one objective is to improve the efficiency in design and implementation of programs to get a better result following their analysis.

Work Related to Multi-unit Bilateral Trade Currently the best bound of $1 - \frac{1}{e}$ is provided by the Generalized Random Quantile Mechanism [4], and there are no hardness results. In this project, we plan to work generalizing the mechanisms in [5] [3] in this setting with multiple identical items, and see whether the bound can be preserved or there are instances leading to hardness results.

1.4 Outline

The remaining of this report proceeds as follows. Chapter 2 explains the flow of analysis in two research papers [5] [3] that are close to this work. Chapter 3 discusses the current work on Fixed-Price Mechanisms. Chapter 4 provides a plan for the work in the next phase on the multiple item setting. Chapter 5 is a conclusion for this report.

Chapter 2

Analysis Methods

The part of this project related to Fixed-Price Mechanisms follows the analysis in two researches [5] [3], this section briefly introduces their analysis and programs.

2.1 Characterizing the optimal price distribution for buyer distribution

In [5], the authors consider the setting that the mechanism is given only F_B. They characterize the optimal price distribution with respect to F_B, and the mechanism posts a price p sampled from this distribution. They prove a lower bound of 0.71 and an upper bound of 0.7381 for this mechanism. Their general idea for proving the lower bound is introduced in this section.

To prove α is a lower bound, it is equivalent to show that:

$$\min_{I=(F_S,F_B)} \max_{F_P} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\} \geq 0 \qquad \forall I$$

We write $\bar{F}_B(b) = 1 - F_B(b)$ as the complementary CDF. The above inequality can be rewritten as:

$$\min_{I=(F_S,F_B)} \max_{F_P}\{\mathbb{E}_p[\mathbb{E}_s[s+\left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b+\bar{F}_B(p)\cdot(p-s)\right)\cdot 1_{s\leq p}]] \\ -\alpha\cdot\mathbb{E}_s[s+\int_s^\infty \bar{F}_B(b)\mathrm{d}b]\}\geq 0 \qquad \forall I$$

Knowing one-sided prior information can never be better than knowing two-sided prior information. A lower bound of the former directly implies a lower bound of the latter. The authors consider the setting that only F_B is given, and show the existence of a F_P with respect to any F_B such that the inequality is satisfied for any F_S. Fix an arbitrary F_B, it is sufficient and necessary to show that there exists a price distribution F_P (CDF) such that for any $s \geq 0$:

$$s+\mathbb{E}_{p\sim F_P}[\left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b+\bar{F}_B(p)\cdot(p-s)\right)\cdot 1_{s\leq p}] \\ -\alpha\cdot\left(s+\int_s^\infty \bar{F}_B(b)\mathrm{d}b\right)\geq 0$$

We write f_p as the PDF. This is further equivalent to:

$$(1-\alpha)s+\int_s^\infty \left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b+\bar{F}_B(p)\cdot(p-s)\right)f_p(p)\mathrm{d}p \\ -\alpha\int_s^\infty \bar{F}_B(b)\mathrm{d}b\geq 0 \qquad \forall s\geq 0$$

f_p should satisfy that (1) it has non-negative densities, (2) the above inequality holds, and (3) $\int_0^\infty f_p(p)\mathrm{d}p=1$.

Consider an optimization problem that minimizes $\int_0^\infty f_p(p)\mathrm{d}p$ subject to constraints (1) and (2). Suppose an f_p' satisfies (1) and (2), and $\int_0^\infty f_p(p)'\mathrm{d}p<1$, we can always find a f_p that satisfies (3) without violating (1) and (2) by adding positive densities at $f_p(0)$. But if $\int_0^\infty f_p(p)'\mathrm{d}p>1$, then there is no way to find a feasible f_p.

Therefore, to prove α is a lower bound is equivalent to show that $\int_0^\infty f_p(p)\mathrm{d}p\leq 1$.

Notice that $(1-\alpha)s$ is strictly increasing in s, and $\alpha \int_s^\infty \bar{F}_B(b)\mathrm{d}b$ is non-increasing in s. There exists a unique $\bar{s}$ such that:

$$(1-\alpha)\bar{s} = \alpha \int_{\bar{s}}^{\infty} \bar{F}_B(b)\mathrm{d}b$$

For any $s \geq \bar{s}$, (2) is satisfied naturally.

The optimal price distribution f_p^* that achieves the minimum of $\int_0^\infty f_p(p)\mathrm{d}p$ should satisfy that:

$$(1-\alpha)s + \int_s^\infty \left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b + \bar{F}_B(p)\cdot(p-s) \right) f_p^*(p)\mathrm{d}p \\ - \alpha \int_s^\infty \bar{F}_B(b)\mathrm{d}b = 0 \qquad \forall 0 \leq s \leq \bar{s}$$

$$f_p^*(s) = 0 \qquad \forall s > \bar{s}$$

That is, for any $0 \leq s \leq \bar{s}$,

$$f_P^*(s) = \alpha \cdot \left(\frac{\bar{F}_B(s)}{\int_s^\infty \bar{F}_B(b)\mathrm{d}b} - \frac{\int_s^{\bar{s}} \bar{F}_B(b)^2\mathrm{d}b}{\left(\int_s^\infty \bar{F}_B(b)\mathrm{d}b\right)^2} \right) + (1-\alpha)\cdot \frac{\int_{\bar{s}}^\infty \bar{F}_B(b)\mathrm{d}b}{\left(\int_s^\infty \bar{F}_B(b)\mathrm{d}b\right)^2}$$

WLOG, assume $\bar{s} = 1$. We then have $\int_1^\infty \bar{F}_B(b)\mathrm{d}b = \frac{1-\alpha}{\alpha}$. To prove α is a lower bound is equivalent to show $\int_0^1 f_p^*(s)\mathrm{d}s \leq 1$.

f_P^* is represented by α and functions of $\bar{F}_B$. It is difficult to find the anti-derivative and calculate $\int_0^1 f_p^*(s)\mathrm{d}s$ directly, the idea is to use discretization and verify the inequality for all non-increasing step functions $\bar{F}_B$ in domain $[0,1]$.

Discretization The step function $\bar{F}_B$ may decrease at any point in $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$, and it is restricted to take values from $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$. We use $\bar{F}_B'$ to denote the function before discretization. On grid points, round it down to the largest feasible value:

$$\bar{F}_B(\frac{i}{n}) = \lfloor \bar{F}_B'(\frac{i}{n}) \cdot n \rfloor \cdot \frac{1}{n}$$

For any $s \in (\frac{i-1}{n}, \frac{i}{n})$, extend the value of $\bar{F}_B(\frac{i}{n})$:

$$\bar{F}_B(s) = \bar{F}_B(\frac{i}{n})$$

$\bar{F}_B(s)^2$ is obtained by taking the square of $\bar{F}_B(s)$. We can get $\int_s^1 \bar{F}_B(b)\mathrm{d}b$ and $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$ directly by their geometric meanings. For any $s \in (\frac{i-1}{n}, \frac{i}{n}]$:

$$\int_s^1 \bar{F}_B(b)\mathrm{d}b = (\frac{i}{n} - s) \cdot \bar{F}_B(\frac{i}{n}) + \sum_{j=i+1}^{n} \bar{F}_B(\frac{j}{n}) \cdot \frac{1}{n}$$

$$\int_s^1 \bar{F}_B(b)^2\mathrm{d}b = (\frac{i}{n} - s) \cdot \bar{F}_B(\frac{i}{n})^2 + \sum_{j=i+1}^{n} \bar{F}_B(\frac{j}{n})^2 \cdot \frac{1}{n}$$

Notice that after discretization, $\bar{F}_B(s)$ can take values from $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$, $\int_s^1 \bar{F}_B(b)\mathrm{d}b$ can take values from $\{0, \frac{1}{n^2}, \frac{2}{n^2}, \cdots, 1\}$, and $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$ can take values from $\{0, \frac{1}{n^3}, \frac{2}{n^3}, \cdots, 1\}$.

Dynamic Programming Algorithm The authors use a Dynamic Programming algorithm to search for the $\bar{F}_B(s)$ that achieves the maximum of $\int_0^1 f_p^*(s)\mathrm{d}s$. We first note that:

$$\int_0^1 f_p^*(s)\mathrm{d}s = \sum_0^{n-1} \int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s$$

Using the discretized $\bar{F}_B(s)$, $\int_s^1 \bar{F}_B(b)\mathrm{d}b$ and $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$, we have:

$$\int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s = \frac{\alpha \left(\bar{F}_B(\frac{i+1}{n}) \int_{(i+1)/n}^1 \bar{F}_B(b)\mathrm{d}b - \int_{(i+1)/n}^1 \bar{F}_B(b)^2\mathrm{d}b\right) + (1-\alpha)^2/\alpha}{\int_{(i+1)/n}^1 \bar{F}_B(b)\mathrm{d}b \left(\bar{F}_B(\frac{i+1}{n}) \int_{(i+1)/n}^1 \bar{F}_B(b)\mathrm{d}b + \frac{1}{n}\bar{F}_B(\frac{i+1}{n})\right)} \cdot \frac{1}{n}$$

$\int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s$ can be represented by α and values that are multiples of $\frac{1}{n}$, $\frac{1}{n^2}$ and $\frac{1}{n^3}$.

We use $Sol_k(x, y, z)$ to denote the objective value of the problem:

$$\begin{aligned} \max \qquad & \int_0^{k/n} f_p^*(s)\mathrm{d}s \\ \text{s.t.} \qquad & x = \bar{F}_B(\frac{k}{n}) \\ & y = \int_{k/n}^{1} \bar{F}_B(b)\mathrm{d}b \\ & z = \int_{k/n}^{1} \bar{F}_B(b)^2\mathrm{d}b \end{aligned}$$

The Dynamic Programming algorithm is presented as follows:

1. Initialize Sol_0 by a $(n+1) \times (n^2+1) \times (n^3+1)$ 3d-array of zeros
2. for k in range $[0, n-1]$:
3.   Initialize Sol_{k+1} by a $(n+1) \times (n^2+1) \times (n^3+1)$ 3d-array of zeros
4.   for each cell in Sol_{k+1}:
5.   if x, y, z form a valid case:
6.   for $x' \leq x$:
7. $y' = y - x'/n$
8. $z' = z - x'^2/n$
9.   if y', $z' \geq 0$:
10.   if $Sol_k(x, y, z) + \int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s > Sol_{k+1}(x', y', z')$:
10. $Sol_{k+1}(x', y', z') = Sol_k(x, y, z) + \int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s$

It remains to find the largest value in the array in the last round, we denote this value as $maxFP$. The values x, y, z can take are restricted, and they form a valid case if it holds that $(1-s)\int_s^1 \bar{F}_B(b)^2\mathrm{d}b \geq \left(\int_s^1 \bar{F}_B(b)\mathrm{d}b\right)^2$ and $\bar{F}_B(s)\int_s^1 \bar{F}_B(b)\mathrm{d}b \geq \int_s^1 \bar{F}_B(b)^2\mathrm{d}b$.

We use $f_p^{**}(s)$ to denote the price distribution w.r.t. $\bar{F}_B'$ before discretization. The discretization error is analyzed as:

$$err = \int_0^1 f_p^{**}(s)\mathrm{d}s - \int_0^1 f_p^*(s)\mathrm{d}s \leq \alpha ln \frac{1-\alpha}{1-\alpha(1+2/n)}$$

To prove a lower bound of α requires $maxFP + err \leq 1$. The authors implement the algorithm in Python 3.9, take $\alpha = 0.71, n = 75$, they prove that 0.71 is a lower bound.

2.2 Characterizing the optimal mechanism under full prior information

In [3], the authors characterize the optimal Fixed-Price Mechanism for any $I = (F_B, F_S)$. They prove a lower bound of 0.72 and an upper bound of 0.7381. This section introduces the program in their proof.

The mechanism posts a price p that maximizes the social welfare in expectation:

$$p \in \underset{p}{\operatorname{argmax}} \frac{\mathbb{E}_p[\mathrm{ALG}(I;p)]}{\mathrm{OPT}(I)}$$

As the instances can be scaled, we can assume WLOG that $\mathrm{OPT}(I) = 1$. The following program is used to capture the worst-case instance for the mechanism:

$$\begin{aligned}
\min_{\alpha, F_B, F_S} \quad & \alpha \\
\text{s.t.} \quad & \mathrm{OPT}(I) := \int_0^\infty \int_0^\infty \max\{s, b\} \mathrm{d}F_B(b) \mathrm{d}F_S(s) \geq 1 \\
& \mathrm{ALG}(I;p) := \int_0^\infty s \mathrm{d}F_S(s) + \int_0^p \int_p^\infty (b - s) \mathrm{d}F_B(b) \mathrm{d}F_S(s) \leq \alpha \qquad \forall p
\end{aligned}$$

The solution α is the tight ratio that can be achieved by the optimal Fixed-Price Mechanism on the worst-case instance F_B, F_S in the solution set.

But this program cannot be solved directly as it has infinite dimensions, the authors discretize the support and restrict the mechanism to choose the best price from this support. They prove that any instance can be discretized and rounded to this finite support such that it holds $discrete - \mathrm{OPT}(I) \geq \mathrm{OPT}(I)$

and *discrete* $-$ ALG$(I;p) \leq$ ALG$(I;p)$. The program with finite variables can be solved by optimization solvers. They choose a specific support and prove an lower bound of 0.72.

Remark In [5] and [3], the authors give two different methods to prove that the mechanisms having access to one-sided prior information and full prior information give the same approximation ratio. One proof idea [3] is using the Minimax Theorem. For any fixed F_B:

$$\begin{aligned}
&\min_{F_B} \max_{F_P} \min_{F_S} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\} \\
&\qquad = \min_{F_B} \min_{F_S} \max_{F_P} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\} \\
&\qquad\qquad = \min_{I=(F_S,F_B)} \max_{F_P} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\}
\end{aligned}$$

Ideally with unlimited computational resources, the lower bound and the upper bound can converge, and the approximation ratio of two prior information settings should be the same.

Chapter 3

Work and Discussion

As introduced in the previous chapter, [5] and [3] use different analysis and programs. The program in [3] has quadratic constraints and quadratic objectives, and is solved using an optimization solver Gurobi. The algorithms the solver use can keep track of the best solution currently found and the bound of the optimal solution, and we can know the gap between the current one and the optimal one. These are not polynomial time algorithms, but there have been extensive research on reducing the running time. The authors choose a specific support for the program and the solver is effective in solving the problem.

The time complexity of the Dynamic Programming algorithm in [5] is $O(n^8)$. In implementation, the authors represent $\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b$ approximately by rounding it to multiples of $\frac{1}{n^2}$ to reduce the time complexity. In this project, we want to be more precise and allow $\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b$ to take multiples of $\frac{1}{n^3}$. Then, we need to implement the algorithm more efficiently to reduce the running time. This section introduces our work in improving the efficiency in implementation and design.

The authors implement the algorithm using Python, and the values in the 3d-array are stored as double, which keeps 16 decimal places. We change the programming language and data type to reduce the running time. For $n = 35$, the algorithm in [5] run for one hour, by directly switching from Python to Java and from double to float (keeps 8 decimal places), the running time is reduced to 8 seconds.

Another adjustment is to reverse the update order. Symmetrically, we use $Sol_k(x, y, z)$ to denote the objective value of the problem:

$$
\begin{aligned}
\max \quad & \int_{1-k/n}^{1} f_p^*(s)\mathrm{d}s \\
\text{s.t.} \quad & x = \bar{F}_B(1 - \frac{k}{n}) \\
& y = \int_{1-k/n}^{1} \bar{F}_B(b)\mathrm{d}b \\
& z = \int_{1-k/n}^{1} \bar{F}_B(b)^2\mathrm{d}b
\end{aligned}
$$

k is in the set $\{1, 2, \cdots, n\}$. Sol_k stores values of $\int_{1-k/n}^{1} f_p^*(s)\mathrm{d}s$ with respect to different x, y, z. The advantage of this adjustment is to make use of the constraints in $\int_{1-1/n}^{1} f_p^*(s)\mathrm{d}s$:

$$
\begin{aligned}
x &= \bar{F}_B(1 - \frac{1}{n}) \geq \bar{F}_B(1 - \frac{1}{n} + \epsilon) \\
y &= \bar{F}_B(1 - \frac{1}{n} + \epsilon) \times \frac{1}{n} \\
z &= \bar{F}_B(1 - \frac{1}{n} + \epsilon)^2 \times \frac{1}{n}
\end{aligned}
$$

In each round, the objective is updated from maximizing $\int_{1-k/n}^{1} f_p^*(s)\mathrm{d}s$ to maximizing $\int_{1-(k+1)/n}^{1} f_p^*(s)\mathrm{d}s$. We have the following dependency:

$$
\begin{aligned}
x' &= \bar{F}_B(1 - \frac{k+1}{n}) \geq x \\
y' &= \int_{1-(k+1)/n}^{1} \bar{F}_B(b)\mathrm{d}b \;=\; y + x \times \frac{1}{n} \\
z' &= \int_{1-(k+1)/n}^{1} \bar{F}_B(b)^2\mathrm{d}b \;=\; z + x^2 \times \frac{1}{n}
\end{aligned}
$$

The structure follows the Dynamic Programming algorithm in [5]. In round n, we have Sol_n that stores the maximum objective values of $\int_0^1 f_p^*(s)\mathrm{d}s$. It remains to find the largest value in Sol_n, we denote it as $maxFP$.

As x can only take values that are multiples of $\frac{1}{n}$ and $x \in [0,1]$, we can know all value combinations of (x,y,z) in calculating $\int_{1-1/n}^{1} f_p^*(s)\mathrm{d}s$. One implementation idea is to use (x,y,z) as the key and $Sol_k(x,y,z)$ as the value, and Sol_k stores the key, value pairs. An efficient implementation in Java is using HashMap. Similar structures in other programming languages include dictionary in Python, and unordered_map in C++. But these implementations do not support multi-threading.

Another implementation is following the authors' design and use 3d-array, which allows multi-threading for speeding up. Note that it always holds:

$$\int_{1-(k+1)/n}^{1-k/n} f_p^*(s)\mathrm{d}s \geq 0$$

The values in the array that are modified to be non-zero have valid combinations of (x,y,z) as indices. But in terms of memory use, array is not as efficient.

A comparison of running time of different implementations is as follows:

Implementation	n=22	n=23	n=24	n=25	n=26
Array_original	12	16	23		
Array_reversed	7	11	15		
HashMap	10	13	20	27	38

Table 3.1: Comparison of running time (s)

As we use higher precision to represent $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$, we notice that main memory space can be a limitation. For $n = 25$ and $n = 26$, there are not enough space in heap memory for array implementation.

This method relies on the discretization of F_B, the discretization error is $err \leq \alpha ln\frac{1-\alpha}{1-\alpha(1+2/n)}$. Our goal is to make sure $maxFP + err \leq 1$. To reduce err to 0.05 requires $n = 72$, for $err \approx 0.03$ requires $n = 125$. Directly following this method to improve the currently best ratio of 0.72 can be challenging because of the excessive need for main memory space.

Chapter 4

Work Plan

The second part of this project from Jan 2025 is on bilateral trade with multiple identical items. In this project, we focus on mechanisms that are Dominant Strategy Incentive Compatible, Individually Rational, and Strongly Budget Balanced. [4] propose the Multi-unit Fixed Price Mechanism, which satisfies the above constraints. The mechanism picks a fixed unit price p and iteratively offers this price to the seller and buyer until one of the agents rejects. They generalize the mechanisms in [2] [1] into the multi-unit setting and prove an approximation of $\frac{1}{2}$ for the Generalized Median Mechanism and an approximation of $1 - \frac{1}{e}$ for the Generalized Random Quantile Mechanism.

Now we know the mechanisms with better lower bounds for the 1 item setting, we want to know what bounds can be achieved if the mechanisms is generalized into the multi-unit setting, or if there are instances that lead to hardness results. The work on the multi-unit setting is to generalize these mechanisms and analyze their performance.

Chapter 5

Conclusion

To conclude, the first part of this project is on Fixed-Price Mechanisms. Following the analysis in [5], we implement the Dynamic Programming algorithm more efficiently in terms of running time and memory use. The difficulties are that reducing the discretization error requires large n. Due to the limitation of the hardware, improving the current bound would require more insights into the problem.

In the second phase, the work is on generalized bilateral trade with multiple identical items. We plan to generalize the mechanisms that give better bounds for the 1 item setting, and analyze the performance on the multiple-item setting.

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