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COMP4801 FINAL REPORT

Fixed-Price Mechanisms in Bilateral Trade

Research on Algorithm Design and Analysis

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Abstract

This project studies social welfare maximization in bilateral trade. The model comprises one seller, one buyer, and one indivisible item. As research shows that no truthful mechanisms can be socially efficient, there has been extensive work on approximating the optimal social welfare. In this project, our work is close to two recent studies that give the best-known approximation ratio using Fixed-Price Mechanisms. We follow their analysis and use more efficient and accurate programs on the problem, and also make improvement in the analysis of discretization error.

Acknowledgments

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Chapter 1

Introduction

The bilateral trade model studies that one buyer and one seller trade an indivisible item. Research shows that no incentive-compatible, individually rational, and budget-balanced mechanism can achieve the optimal social welfare [6]. Motivated by this impossibility result, many researchers have been working on approximately efficient mechanisms. The Fixed-Price Mechanism posts a price p, and trade occurs if both agents accept this price. Two recent studies respectively show that there exists a fixed price mechanism that guarantees 0.71 [5] and 0.72 [3] fraction of the optimal social welfare for any buyer and seller value distributions, and no fixed price mechanism can achieve more than 0.7381 fraction of the optimal welfare. Generalizations of the bilateral trade model such as Multi-unit Bilateral Trade have also been studied. [4] shows that there exists an incentive-compatible, individually rational, and budget-balanced mechanism which achieves a $(1 - \frac{1}{e})$-approximation.

The work of this project is closely related to [5]. We follow their analysis and try to improve the efficiency and precision of the programs. And we provide a different approach to analyze the error bound. This project also considers generalizing this mechanism into the multi-unit setting, but currently there is no result to describe the performance of the generalized mechanism.

1.1 Background

This section provides relevant background information on the Bilateral Trade Model, Fixed-Price Mechanisms, approximation algorithms, and the Multi-unit Bilateral Trade Model.

Bilateral Trade Model In the bilateral trade model, a seller owns an indivisible item, and a buyer wants to purchase it. Both agents have private valuations of the item, we denote the seller's valuation as s and the buyer's valuation as b. We only know the public probability distribution (cumulative distribution functions) from which the valuations are drawn, i.e., $s \sim F_S$, $b \sim F_B$. A socially efficient mechanism always trades when the buyer has higher values, and never trades if the seller has higher values.

Fixed-Price Mechanisms The Fixed-Price Mechanism posts p as the price. Individual Rationality requires that $s \leq p \leq b$, that is, both agents accept the price if and only if they expect non-negative payoffs. In some cases, agents may lie about their valuations in order to gain more benefits from trade. The Fixed-Price Mechanism is Dominant Strategy Incentive Compatible, agents have no incentive to deviate from using the strategy of their true valuations. This mechanism is also Strongly Budget Balanced, the payment of the buyer is fully transferred to the seller.

Social Welfare and Approximation Ratio The social welfare is b if the trade occurs and s otherwise. The optimal social welfare is defined as $\mathrm{OPT}(F_S, F_B) = \mathbb{E}_{s\sim F_S, b\sim F_B}[b + (b - s) \cdot 1_{s\leq b}]$. The welfare of the Fixed-Price Mechanism using p as the price is $\mathrm{ALG}(F_S, F_B; p) = \mathbb{E}_{s\sim F_S, b\sim F_B}[b + (b - s) \cdot 1_{s\leq p\leq b}]$.

The approximation ratio describes the performance of the mechanism on the worst-case instance:

$$\alpha = \min_{I=(F_S,F_B)} \frac{\mathrm{ALG}(I; p)}{\mathrm{OPT}(I)}$$

To maximize the above ratio is equivalent to maximize the welfare obtained by the mechanism.

Multi-unit Bilateral Trade Model The Multi-unit Bilateral Trade Model studies one buyer and one seller who initially holds k identical items. Both agents have private increasing submodular valuation functions, reflecting to what extent they value the holding of the number of units. Let v_s denote the valuation function of the seller, drawn from public G_S, and let v_b denote the valuation function of the buyer, drawn from public G_B. An increasing submodular valuation function v satisfies that for any $x, y \in [k]$ and $x < y$, $v(x) < v(y)$, $v(x) - v(x-1) \geq v(y) - v(y-1)$. This captures a common economic phenomenon that owning one more unit is always desirable, but the marginal utility of one more unit decreases as the held amount goes up. An instance is a pair (G_S, G_B, k). The mechanism M needs to decide the number of units q transferred from the seller to the buyer and the payment transferred from the buyer to the seller.

1.2 Related research

In 1983, Myerson and Satterthwaite [6] initiated the study on the bilateral trade model. Their work shows that as long as the buyer and seller distributions are continuous with positive probability densities and the value distribution intervals overlap, no truthful mechanism can be socially efficient. Many researchers have been working on exploring what is possible.

Some fixed-price mechanisms are efficient in approximating social welfare. The Median Mechanism [2] sets the median of the seller's distribution as the price. This mechanism provides a $\frac{1}{2}$ approximation. The Random Quantile Mechanism [1] chooses a quantile x randomly from a certain distribution and outputs the x-quantile of the seller distribution as the price. This mechanism is an $1 - \frac{1}{e}$ approximation. The Optimal Fixed Price Mechanism [5] [3] posts a price that maximizes the welfare of the mechanism. The approximation ratio is around 0.71 and 0.72 respectively. In terms of using fixed-price mechanisms to approximate welfare, this mechanism is the most efficient.

[4] is on the Multi-unit Bilateral Trade Model. They show Generalized Median Mechanism provides a $\frac{1}{2}$ approximation and Generalized Random Quantile Mechanism is a $1 - \frac{1}{e}$ approximation, preserving the ratio of the single-item setting.

1.3 Motivation and objectives

Work Related to Single-Item Fixed-Price Mechanisms Two papers [5] [3] on Fixed-Price Mechanisms that give the best known bound of 0.71 and 0.72 both apply the method of numerically solving a program to approach the approximation ratio, leaving a relatively small gap from the impossibility result of 0.7381. They also state in the paper that the lower bound and the upper bound can converge with more computational resources. In this project, the objective is to improve the efficiency in design and implementation of programs to get a better result following their analysis.

Work Related to Multi-unit Bilateral Trade Currently the best bound of $1 - \frac{1}{e}$ is provided by the Generalized Random Quantile Mechanism [4], and there are no hardness results. In this project, we plan to work generalizing the mechanisms in [5] to the setting with multiple identical items, and study the performance of the generalized mechanism on the multi-unit setting.

1.4 Outline

The remaining of this report proceeds as follows. Chapter 2 explains the flow of analysis in two research papers [5] [3] that are close to this work. Chapter 3 discusses the work on single-item Fixed-Price Mechanisms. Chapter 4 discusses the work on the multi-unit setting and challenges. Chapter 5 is a conclusion for this report.

Chapter 2

Analysis Methods

The part of this project related to single-item Fixed-Price Mechanisms are close to two researches [5] [3], this section briefly introduces their analysis and programs.

2.1 Characterizing the optimal price distribution for buyer distribution

In [5], the authors consider the setting that the mechanism is given only F_B. They characterize the optimal price distribution with respect to F_B, and the mechanism posts a price p sampled from this distribution. They prove a lower bound of 0.71 and an upper bound of 0.7381 for this mechanism. Their general idea for proving the lower bound is introduced in this section.

2.1.1 Preliminaries and characterizations

To prove α is a lower bound, it is equivalent to show that:

$$\min_{I=(F_S,F_B)} \max_{F_P} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\} \geq 0 \qquad \forall I$$

We write $\bar{F}_B(b) = 1 - F_B(b)$ as the complementary cumulative distribution function. The above inequality can be rewritten as:

$$\min_{I=(F_S,F_B)} \max_{F_P}\{\mathbb{E}_p[\mathbb{E}_s[s + \left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b + \bar{F}_B(p)\cdot(p-s)\right)\cdot 1_{s\le p}]] - \alpha\cdot\mathbb{E}_s[s + \int_s^\infty \bar{F}_B(b)\mathrm{d}b]\} \ge 0 \qquad \forall I$$

Knowing one-sided prior information can never be better than knowing two-sided prior information. A lower bound of the former directly implies a lower bound of the latter. The authors consider the setting that an arbitrary F_B is given, and show the existence of a F_P with respect to F_B such that the inequality is satisfied for any F_S. Fix an arbitrary F_B, it is sufficient and necessary to show that there exists a price distribution F_P (cumulative distribution function) such that for any $s \ge 0$:

$$s + \mathbb{E}_{p\sim F_P}[\left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b + \bar{F}_B(p)\cdot(p-s)\right)\cdot 1_{s\le p}] - \alpha\cdot\left(s + \int_s^\infty \bar{F}_B(b)\mathrm{d}b\right) \ge 0$$

We write f_p as the probability density function. This is further equivalent to:

$$(1-\alpha)s + \int_s^\infty \left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b + \bar{F}_B(p)\cdot(p-s)\right) f_p(p)\mathrm{d}p - \alpha\int_s^\infty \bar{F}_B(b)\mathrm{d}b \ge 0 \qquad \forall s \ge 0$$

f_p should satisfy that (1) it has non-negative densities, (2) the above inequality holds, and (3) $\int_0^\infty f_p(p)\mathrm{d}p = 1$.

Consider an optimization problem that minimizes $\int_0^\infty f_p(p)\mathrm{d}p$ subject to constraints (1) and (2). Suppose an f_p° satisfies (1) and (2), and $\int_0^\infty f_p^\circ(p)\mathrm{d}p < 1$, we can always find a f_p that satisfies (3) without violating (1) and (2) by adding positive densities at $f_p(0)$. But if $\int_0^\infty f_p^\circ(p)\mathrm{d}p > 1$, then there is no way to find a feasible f_p.

Therefore, to prove α is a lower bound is equivalent to show that $\int_0^\infty f_p(p)\mathrm{d}p \leq 1$.

Notice that $(1-\alpha)s$ is strictly increasing in s, and $\alpha \int_s^\infty \bar{F}_B(b)\mathrm{d}b$ is non-increasing in s. There exists a unique $\bar{s}$ such that:

$$(1-\alpha)\bar{s} = \alpha \int_{\bar{s}}^{\infty} \bar{F}_B(b)\mathrm{d}b$$

For any $s \geq \bar{s}$, (2) is satisfied naturally.

The optimal price distribution f_p^* that achieves the minimum of $\int_0^\infty f_p(p)\mathrm{d}p$ should satisfy that:

$$(1-\alpha)s + \int_s^\infty \left(\int_p^\infty \bar{F}_B(b)\mathrm{d}b + \bar{F}_B(p) \cdot (p-s) \right) f_p^*(p)\mathrm{d}p \\ - \alpha \int_s^\infty \bar{F}_B(b)\mathrm{d}b = 0 \qquad \forall 0 \leq s \leq \bar{s}$$

$$f_p^*(s) = 0 \qquad \forall s > \bar{s}$$

This means, for any $0 \leq s \leq \bar{s}$,

$$f_P^*(s) = \alpha \cdot \left(\frac{\bar{F}_B(s)}{\int_s^\infty \bar{F}_B(b)\mathrm{d}b} - \frac{\int_s^{\bar{s}} \bar{F}_B(b)^2\mathrm{d}b}{\left(\int_s^\infty \bar{F}_B(b)\mathrm{d}b\right)^2} \right) + (1-\alpha) \cdot \frac{\int_{\bar{s}}^\infty \bar{F}_B(b)\mathrm{d}b}{\left(\int_s^\infty \bar{F}_B(b)\mathrm{d}b\right)^2}$$

Without loss of generality, assume $\bar{s} = 1$. We then have $\int_1^\infty \bar{F}_B(b)\mathrm{d}b = \frac{1-\alpha}{1-\alpha}$.[1] To prove α is a lower bound is equivalent to showing $\int_0^1 f_p^*(s)\mathrm{d}s \leq 1$.

f_P^* is represented by α and functions of $\bar{F}_B$. It is difficult to find the anti-derivative and calculate $\int_0^1 f_p^*(s)\mathrm{d}s$ directly, the idea is to use discretization and verify the inequality for all non-increasing step functions $\bar{F}_B$ in domain $[0,1]$.

2.1.2 Discretization and Dynamic Programming Algorithm

The step function $\bar{F}_B$ may decrease at any point in $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$, and it is restricted to take values from $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$. We use $\bar{F}_B^*$ to denote the function

[1] This would require $\bar{F}_B(1) > 0$, but we can allow $\bar{F}_B(1)$ to be 0 by considering the extreme case that there is a value $1 + \frac{1-\alpha}{\alpha\epsilon}$ with probability ϵ for an extremely small $\epsilon > 0$.

before discretization. On grid points, round it down to the largest feasible value:

$$\bar{F}_B(\frac{i}{n}) = \lfloor \bar{F}_B^*(\frac{i}{n}) \cdot n \rfloor \cdot \frac{1}{n}$$

For any $s \in (\frac{i-1}{n}, \frac{i}{n})$, extend the value of $\bar{F}_B(\frac{i}{n})$:

$$\bar{F}_B(s) = \bar{F}_B(\frac{i}{n})$$

$\bar{F}_B(s)^2$ is obtained by taking the square of $\bar{F}_B(s)$. We can get $\int_s^1 \bar{F}_B(b)\mathrm{d}b$ and $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$ directly by their geometric meanings. For any $s \in (\frac{i-1}{n}, \frac{i}{n}]$:

$$\int_s^1 \bar{F}_B(b)\mathrm{d}b = (\frac{i}{n} - s) \cdot \bar{F}_B(\frac{i}{n}) + \sum_{j=i+1}^{n} \bar{F}_B(\frac{j}{n}) \cdot \frac{1}{n}$$

$$\int_s^1 \bar{F}_B(b)^2\mathrm{d}b = (\frac{i}{n} - s) \cdot \bar{F}_B(\frac{i}{n})^2 + \sum_{j=i+1}^{n} \bar{F}_B(\frac{j}{n})^2 \cdot \frac{1}{n}$$

Notice that after discretization, $\bar{F}_B(s)$ can take values from $\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$, $\int_s^1 \bar{F}_B(b)\mathrm{d}b$ can take values from $\{0, \frac{1}{n^2}, \frac{2}{n^2}, \cdots, 1\}$, and $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$ can take values from $\{0, \frac{1}{n^3}, \frac{2}{n^3}, \cdots, 1\}$.

The authors use a Dynamic Programming algorithm to search for the $\bar{F}_B(s)$ that achieves the maximum of $\int_0^1 f_p^*(s)\mathrm{d}s$. We first note that:

$$\int_0^1 f_p^*(s)\mathrm{d}s = \sum_0^{n-1} \int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s$$

Using the discretized $\bar{F}_B(s)$, $\int_s^1 \bar{F}_B(b)\mathrm{d}b$ and $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$, we have:

$$\int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s = \frac{\alpha\left(\bar{F}_B(\frac{i+1}{n})\int_{(i+1)/n}^1 \bar{F}_B(b)\mathrm{d}b - \int_{(i+1)/n}^1 \bar{F}_B(b)^2\mathrm{d}b\right) + (1-\alpha)^2/\alpha}{\int_{(i+1)/n}^1 \bar{F}_B(b)\mathrm{d}b\left(\bar{F}_B(\frac{i+1}{n})\int_{(i+1)/n}^1 \bar{F}_B(b)\mathrm{d}b + \frac{1}{n}\bar{F}_B(\frac{i+1}{n})\right)} \cdot \frac{1}{n}$$

$\int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s$ can be represented by α and values that are multiples of $\frac{1}{n}$, $\frac{1}{n^2}$ and $\frac{1}{n^3}$.

We use $Sol_k(x, y, z)$ to denote the objective value of the problem:

$$\begin{aligned} \max \quad & \int_0^{k/n} f_p^*(s)\mathrm{d}s \\ \text{s.t.} \quad & x = \bar{F}_B(\frac{k}{n}) \\ & y = \int_{k/n}^{1} \bar{F}_B(b)\mathrm{d}b \\ & z = \int_{k/n}^{1} \bar{F}_B(b)^2\mathrm{d}b \end{aligned}$$

The original Dynamic Programming algorithm is presented as follows:

Initialize Sol_0 by a $(n+1) \times (n^2+1) \times (n^3+1)$ 3d-array of zeros
for k in range $[0, n-1]$:
    Initialize Sol_{k+1} by a $(n+1) \times (n^2+1) \times (n^3+1)$ 3d-array of zeros
    for each cell in Sol_k:
        if x, y, z form a valid case:
            for $x' \leq x$:
                $y' = y - x'/n$
                $z' = z - x'^2/n$
                if y', $z' \geq 0$:
                    if $Sol_k(x, y, z) + \int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s > Sol_{k+1}(x', y', z')$:
                        $Sol_{k+1}(x', y', z') = Sol_k(x, y, z) + \int_{i/n}^{(i+1)/n} f_p^*(s)\mathrm{d}s$

It remains to find the largest value in the array in the last round, we denote this value as $maxFP$. The values x, y, z can take are restricted, and they form a valid case if it holds that $(1-s)\int_s^1 \bar{F}_B(b)^2\mathrm{d}b \geq \left(\int_s^1 \bar{F}_B(b)\mathrm{d}b\right)^2$ and $\bar{F}_B(s)\int_s^1 \bar{F}_B(b)\mathrm{d}b \geq \int_s^1 \bar{F}_B(b)^2\mathrm{d}b$.[2]

We use $f_p^{**}(s)$ to denote the price distribution with respect to $\bar{F}_B^*$ before discretization.

[2]The first inequality follows from Cauchy-Schwarz Inequality. The second inequality is due to: $\frac{d}{ds}(\bar{F}_B(s)\int_s^1 \bar{F}_B(b)\mathrm{d}b) \leq \frac{d}{ds}(\int_s^1 \bar{F}_B(b)^2\mathrm{d}b) \leq 0$, and $\bar{F}_B(1)\int_1^1 \bar{F}_B(b)\mathrm{d}b = \int_1^1 \bar{F}_B(b)^2\mathrm{d}b$.

The discretization error is analyzed as:

$$err = \int_0^1 f_p^{**}(s)\mathrm{d}s - \int_0^1 f_p^*(s)\mathrm{d}s \leq \alpha ln \frac{1-\alpha}{1-\alpha(1+2/n)}$$

To prove a lower bound of α requires $maxFP + err \leq 1$.

In implementation, the authors round each $\int_{\frac{i}{n}}^1 \bar{F}_B(b)^2 \mathrm{d}b$ down to multiples of $\frac{1}{n^2}$. They implement the algorithm in Python 3.9, take $\alpha = 0.71, n = 75$, they prove that 0.71 is a lower bound.

2.2 Characterizing the optimal mechanism under full prior information

In [3], the authors characterize the optimal Fixed-Price Mechanism for any $I = (F_B, F_S)$. They prove a lower bound of 0.72 and an upper bound of 0.7381. This section introduces the program in their proof.

The mechanism posts a price p that maximizes the social welfare in expectation:

$$p \in \operatorname*{argmax}_p \frac{\mathbb{E}_p[\mathrm{ALG}(I;p)]}{\mathrm{OPT}(I)}$$

As the instances can be scaled, we can assume without loss of generality that $\mathrm{OPT}(I) = 1$. The following program is used to capture the worst-case instance for the mechanism:

$$\begin{aligned}
\min_{\alpha, F_B, F_S} \quad & \alpha \\
\text{s.t.} \quad & \mathrm{OPT}(I) := \int_0^\infty \int_0^\infty \max\{s, b\} \mathrm{d}F_B(b)\mathrm{d}F_S(s) \geq 1 \\
& \mathrm{ALG}(I;p) := \int_0^\infty s\mathrm{d}F_S(s) + \int_0^p \int_p^\infty (b-s)\mathrm{d}F_B(b)\mathrm{d}F_S(s) \leq \alpha \qquad \forall p
\end{aligned}$$

The solution α is the tight ratio that can be achieved by the optimal Fixed-Price Mechanism on the worst-case instance F_B, F_S in the solution set.

But this program cannot be solved directly as it has infinite dimensions, the authors discretize the support and restrict the mechanism to choose the best price from this support. They prove that any instance can be discretized and rounded to this finite support such that it holds $discrete - \mathrm{OPT}(I) \geq \mathrm{OPT}(I)$ and $discrete - \mathrm{ALG}(I;p) \leq \mathrm{ALG}(I;p)$. The program with finite variables can be solved by optimization solvers. They choose a specific support and prove an lower bound of 0.72.

Remark In [5] and [3], the authors give two different methods to prove that the mechanisms having access to one-sided prior information and full prior information give the same approximation ratio. One proof idea [3] is using the Minimax Theorem. For any fixed F_B:

$$\begin{aligned}
&\min_{F_B} \max_{F_P} \min_{F_S} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\} \\
&\qquad = \min_{F_B} \min_{F_S} \max_{F_P} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\} \\
&\qquad\qquad = \min_{I=(F_S,F_B)} \max_{F_P} \{\mathbb{E}_p[\mathrm{ALG}(I;p) - \alpha \cdot \mathrm{OPT}(I)]\}
\end{aligned}$$

Ideally with unlimited computational resources, the lower bound and the upper bound can converge, and the approximation ratio of two prior information settings should be the same.

Chapter 3

Work and Discussion: Single-Item Setting

As introduced in the previous chapter, [5] and [3] use different analysis and programs. The program in [3] has quadratic constraints and quadratic objectives, and is solved using an optimization solver Gurobi. The algorithms the solver use can keep track of the best solution currently found and the bound of the optimal solution, and we can know the gap between the current one and the optimal one. These are not polynomial time algorithms, but there have been extensive research on reducing the running time. The authors choose a specific support for the program and the solver is effective in solving the problem.

The time complexity of the Dynamic Programming algorithm in [5] is $O(n^8)$. In implementation, the authors represent $\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b$ approximately by rounding it down to multiples of $\frac{1}{n^2}$ to reduce the time complexity to $O(n^7)$. In this project, we want to be more precise and to implement the algorithm more efficiently to reduce the running time. This section introduces our work in improving the efficiency in implementation and design.

3.1 Adjustments in design and implementation

The authors implement the algorithm using Python, and the values in the 3d-array are stored as double, which keeps 16 decimal places. We change the programming language and data type to reduce the running time. For $n = 35$, the algorithm in [5] run for one hour, by directly switching from Python to Java and from double to float (keeps 8 decimal places), the running time is reduced to 8 seconds.

Another adjustment is to reverse the update order. Similarly, we use $Sol_i(x, y, z)$ to denote the objective value of the problem:

$$\begin{aligned}
\max \quad & \int_{\frac{i}{n}}^{1} f_p^*(s)\mathrm{d}s \\
\text{s.t.} \quad & x = \bar{F}_B(s) \qquad s \in (\frac{i}{n}, \frac{i+1}{n}) \\
& y = \int_{\frac{i}{n}}^{1} \bar{F}_B(b)\mathrm{d}b \\
& z = \int_{\frac{i}{n}}^{1} \bar{F}_B(b)^2\mathrm{d}b
\end{aligned}$$

i is in the set $\{0, 1, \cdots, n-1\}$. Sol_i stores values of $\int_{\frac{i}{n}}^{1} f_p^*(s)\mathrm{d}s$ with respect to different x, y, z. The advantage of this adjustment is to make use of the constraints for $s \in (1 - \frac{1}{n}, 1]$:

$$\begin{aligned}
\bar{F}_B(s) &\in \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\} \\
\int_{1-1/n}^{1} \bar{F}_B(b)\mathrm{d}b &= \bar{F}_B(s) \times \frac{1}{n} \\
\int_{1-1/n}^{1} \bar{F}_B(b)^2\mathrm{d}b &= \bar{F}_B(s)^2 \times \frac{1}{n}
\end{aligned}$$

In each round, the objective is updated from maximizing $\int_{\frac{i}{n}}^{1} f_p^*(s)\mathrm{d}s$ to maximizing

$\int_{\frac{i-1}{n}}^{1} f_p^*(s)\mathrm{d}s$. We have the following dependency:

$$
\begin{aligned}
x' &= \bar{F}_B(s) \geq \ x \qquad s \in (\frac{i-1}{n}, \frac{i}{n}) \\
y' &= \int_{\frac{i-1}{n}}^{1} \bar{F}_B(b)\mathrm{d}b \ = \ y + x' \times \frac{1}{n} \\
z' &= \int_{\frac{i-1}{n}}^{1} \bar{F}_B(b)^2\mathrm{d}b \ = \ z + (x')^2 \times \frac{1}{n}
\end{aligned}
$$

The structure follows the Dynamic Programming algorithm in [5]. In round n, we have Sol_0 that stores the maximum objective values of $\int_0^1 f_p^*(s)\mathrm{d}s$. It remains to find the largest value in Sol_0, we denote it as $maxFP$.

As x can only take values that are multiples of $\frac{1}{n}$ and $x \in [0,1]$, we can know all value combinations of (x, y, z) in calculating $\int_{1-1/n}^{1} f_p^*(s)\mathrm{d}s$. One implementation idea is to use (x, y, z) as the key and $Sol_i(x, y, z)$ as the value, and Sol_i stores the (key, value) pairs. Efficient data structures for this implementation include unordered_map in C++, HashMap in Java, and dictionary in Python. But these implementations do not directly support multi-threading.

Another implementation is following the authors' design and using the 3d-array, which allows multi-threading for speeding up. Note that it always holds:

$$
\int_{\frac{i}{n}}^{\frac{i+1}{n}} f_p^*(s)\mathrm{d}s \geq 0
$$

The values in the array that are modified to be non-zero have valid combinations of (x, y, z) as indices. But in terms of memory use, array is not as efficient.

To increase the precision, we allow $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$ to take multiples of $\frac{1}{n^3}$. A comparison of running time of different implementations is as follows:

Implementation	n=23	n=24	n=25	n=27	n=29
Array_original	21	31			
Array_reversed	16	23			
HashMap	6	9	12	26	50

Table 3.1: Comparison of running time (s)

As we use a higher precision to represent $\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b$, we notice that main memory space can be a limitation. For $n \geq 25$, there is not enough space in heap memory for array implementation.

This method relies on the discretization of F_B, the discretization error is analyzed as $err \leq \alpha ln \frac{1-\alpha}{1-\alpha(1+2/n)}$. Our goal is to make sure $maxFP + err \leq 1$. Reducing err to 0.05 requires $n = 72$, and $err \approx 0.03$ requires $n = 125$. Directly representing $\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b$ exactly leads to the excessive need for main memory space.

We then use a more accurate way to round $\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b$ down such that the memory is used more efficiently:

$$v = max\{\frac{\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b}{1/n^3}/n, 1\}$$
$$\text{rounded}(\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b) = \lfloor \frac{\int_s^1 \bar{F}_B(b)^2 \mathrm{d}b}{1/n^3}/v \rfloor \times v$$

3.2 Alternative analysis of discretization error

Reducing the error bound can also help to prove a better result as we can increase the lower bound α in the program. In [5], the authors work out the error bound err which is valid for all step functions. Another option is to work out an error bound for each step function $\bar{F}_B$, and verify $\int_0^1 f_p^*(s) + err(\bar{F}_B) \leq 1$ for each $\bar{F}_B$. As we can use more parameters from each $\bar{F}_B$, experiment results show that the error bound can be reduced to around $60\% - 65\%$ of the original.

In this section, let $\bar{F}_B^*$ denote the buyer distribution before discretization, and let $\bar{F}_B$ denote the step function after discretization.

We first show that for $s \in (\frac{i}{n}, \frac{i+1}{n})$,

$$\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b - \int_s^\infty \bar{F}_B(b)\mathrm{d}b \leq \frac{1}{n} + \frac{n-i}{n^2}$$

Proof. We directly have $\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b - \int_s^\infty \bar{F}_B(b)\mathrm{d}b \leq \int_{\frac{i}{n}}^\infty \bar{F}_B^*(b)\mathrm{d}b - \int_{\frac{i}{n}}^\infty \bar{F}_B(b)\mathrm{d}b$. It remains to show $\int_{\frac{i}{n}}^\infty \bar{F}_B^*(b)\mathrm{d}b - \int_{\frac{i}{n}}^\infty \bar{F}_B(b)\mathrm{d}b \leq \frac{1}{n} + \frac{n-i}{n^2}$.

Consider rounding $\bar{F}_B^*(s)$ up and construct another rounded step function $\widehat{\bar{F}_B}(s)$. On grid points, round up to the smallest multiple of $\frac{1}{n}$:

$$\widehat{\bar{F}_B}(\frac{i}{n}) = \lceil \bar{F}_B^*(\frac{i}{n}) \cdot n \rceil \cdot \frac{1}{n} \qquad \forall i \in \{0, 1, \cdots, n-1\}$$

For $s \in (\frac{i}{n}, \frac{i+1}{n})$, extend the value of $\bar{F}_B(\frac{i}{n})$:

$$\widehat{\bar{F}_B}(s) = \widehat{\bar{F}_B}(\frac{i}{n}) \qquad \forall s \in (\frac{i}{n}, \frac{i+1}{n})$$

We have $\int_{\frac{i}{n}}^{\infty} \widehat{\bar{F}_B}(b)\mathrm{d}b - \int_{\frac{i}{n}}^{\infty} \bar{F}_B(b)\mathrm{d}b \leq \frac{1}{n} + \frac{n-i}{n^2}$:

$$\begin{aligned}
\int_{\frac{i}{n}}^{\infty} \widehat{\bar{F}_B}(b)\mathrm{d}b - \int_{\frac{i}{n}}^{\infty} \bar{F}_B(b)\mathrm{d}b &= \int_{\frac{i}{n}}^{1} \widehat{\bar{F}_B}(b)\mathrm{d}b - \int_{\frac{i}{n}}^{1} \bar{F}_B(b)\mathrm{d}b \\
&= \sum_{j=i}^{n-1} \frac{1}{n} \cdot \left(\widehat{\bar{F}_B}(\frac{j}{n}) - \bar{F}_B(\frac{j+1}{n}) \right) \\
&= \sum_{j=i}^{n-1} \frac{1}{n} \cdot \left(\left(\widehat{\bar{F}_B}(\frac{j}{n}) - \widehat{\bar{F}_B}(\frac{j+1}{n}) \right) + \left(\widehat{\bar{F}_B}(\frac{j+1}{n}) - \bar{F}_B(\frac{j+1}{n}) \right) \right) \\
&\leq \frac{1}{n} \cdot \left(\widehat{\bar{F}_B}(\frac{i}{n}) - \widehat{\bar{F}_B}(1) \right) + (n-i) \cdot \frac{1}{n^2} \\
&\leq \frac{1}{n} + \frac{n-i}{n^2}
\end{aligned}$$

□

Next, we analyze the error bound. Let $obj(\bar{F}_B^*)$ denote the objective value of $\int_0^1 f_p^*(s)\mathrm{d}s$ before rounding, and let $obj(\bar{F}_B)$ denote the objective value after rounding. Also, in implementation, we round $\int_s^1 \bar{F}_B(b)^2\mathrm{d}b$ down, denote the rounded function by $K(s)$. For the discretization error, we have:

$$obj(\bar{F}_B^*) - obj(\bar{F}_B) = \alpha \int_0^1 \left(\frac{\bar{F}_B^*(s)}{\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b} - \frac{\bar{F}_B(s)}{\int_s^\infty \bar{F}_B(b)\mathrm{d}b} \right) \mathrm{d}s \qquad (\gamma_1)$$

$$+\alpha \int_0^1 \left(\frac{K(s)}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{\int_s^1 \bar{F}_B^*(b)^2\mathrm{d}b}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s \qquad (\gamma_2)$$

$$+\frac{(1-\alpha)^2}{\alpha} \int_0^1 \left(\frac{1}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} \right) \mathrm{d}s \qquad (\gamma_3)$$

For γ_1, follow the analysis in [5]:

$$\gamma_1 = \alpha \left(ln(\int_0^\infty \bar{F}_B^*(b)\mathrm{d}b) - ln(\int_0^\infty \bar{F}_B(b)\mathrm{d}b) \right)$$
$$\leq \alpha \left(ln(\int_0^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2}{n}) - ln(\int_0^\infty \bar{F}_B(b)\mathrm{d}b) \right)$$

For $\gamma_2 + \gamma_3$, consider the summation of smaller intervals:

$$\gamma_2 + \gamma_3 = \sum_{i=0}^{i=n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} e(s)\mathrm{d}s$$

For any subinterval, use a similar analysis to [5]:

$$\int_{\frac{i}{n}}^{\frac{i+1}{n}} e(s)\mathrm{d}s = \alpha \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{K(s)}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{\int_s^1 \bar{F}_B^*(b)^2\mathrm{d}b}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s$$
$$- \frac{(1-\alpha)^2}{\alpha} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s$$
$$\leq \alpha \int_{\frac{i}{n}}^{\frac{i+1}{n}} K(s) \left(\frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s$$
$$- \frac{(1-\alpha)^2}{\alpha} \int_{\frac{i}{n}}^{\frac{i+1}{n}} K(s) \left(\frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s$$
$$= (\alpha - \frac{(1-\alpha)^2}{\alpha}) \int_{\frac{i}{n}}^{\frac{i+1}{n}} K(s) \left(\frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s$$
$$\leq (\alpha - \frac{(1-\alpha)^2}{\alpha}) \int_{\frac{i}{n}}^{\frac{i+1}{n}} (\int_s^1 \bar{F}_B(b)^2\mathrm{d}b) \left(\frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b)^2} \right) \mathrm{d}s$$
$$\leq (\alpha - \frac{(1-\alpha)^2}{\alpha}) \int_{\frac{i}{n}}^{\frac{i+1}{n}} (\bar{F}_B(s) \int_s^1 \bar{F}_B(b)\mathrm{d}b) \left(\frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b)^2} - \frac{1}{(\int_s^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2n-i}{n^2})^2} \right) \mathrm{d}s$$

The first inequality is due to $K(s) \leq \int_s^1 \bar{F}_B^*(b)^2 \mathrm{d}b$, $0 \leq K(s) \leq 1$, and $\int_s^\infty \bar{F}_B(b)\mathrm{d}b \leq \int_s^\infty \bar{F}_B^*(b)\mathrm{d}b$. The last two inequalities are due to $K(s) \leq \int_s^1 \bar{F}_B(b)^2\mathrm{d}b \leq \bar{F}_B(s)\int_s^1 \bar{F}_B(b)\mathrm{d}b$, and $\int_s^\infty \bar{F}_B^*(b)\mathrm{d}b \leq \int_s^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2n-i}{n^2}$. Next apply integration by substitution.

Let $x := \int_s^1 \bar{F}_B(b)\mathrm{d}b$, and $a := \frac{1-\alpha}{\alpha} + \frac{2n-i}{n^2}$, we have:

$$
\begin{aligned}
\int_{\frac{i}{n}}^{\frac{i+1}{n}} e(s)\mathrm{d}s &\leq (\alpha - \frac{(1-\alpha)^2}{\alpha}) \int_{\int_{\frac{i+1}{n}}^1 \bar{F}_B(b)\mathrm{d}b}^{\int_{\frac{i}{n}}^1 \bar{F}_B(b)\mathrm{d}b} \left(\frac{x}{(x+\frac{1-\alpha}{\alpha})^2} - \frac{x}{(x+a)^2} \right) \mathrm{d}x \\
&= (\alpha - \frac{(1-\alpha)^2}{\alpha}) \left(ln(x + \frac{1-\alpha}{\alpha}) + \frac{\frac{1-\alpha}{\alpha}}{x+\frac{1-\alpha}{\alpha}} - ln(x+a) - \frac{a}{x+a} \right) \Bigg|_{\int_{\frac{i+1}{n}}^1 \bar{F}_B(b)\mathrm{d}b}^{\int_{\frac{i}{n}}^1 \bar{F}_B(b)\mathrm{d}b} \\
&= (\alpha - \frac{(1-\alpha)^2}{\alpha}) \Bigg(ln(\int_{\frac{i}{n}}^\infty \bar{F}_B(b)\mathrm{d}b) + \frac{\frac{1-\alpha}{\alpha}}{\int_{\frac{i}{n}}^\infty \bar{F}_B(b)\mathrm{d}b} - ln(\int_{\frac{i}{n}}^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2n-i}{n^2}) \\
&\qquad - \frac{\frac{1-\alpha}{\alpha} + \frac{2n-i}{n^2}}{\int_{\frac{i}{n}}^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2n-i}{n^2}} - ln(\int_{\frac{i+1}{n}}^\infty \bar{F}_B(b)\mathrm{d}b) - \frac{\frac{1-\alpha}{\alpha}}{\int_{\frac{i+1}{n}}^\infty \bar{F}_B(b)\mathrm{d}b} \\
&\qquad + ln(\int_{\frac{i+1}{n}}^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2n-i}{n^2}) + \frac{\frac{1-\alpha}{\alpha} + \frac{2n-i}{n^2}}{\int_{\frac{i+1}{n}}^\infty \bar{F}_B(b)\mathrm{d}b + \frac{2n-i}{n^2}} \Bigg)
\end{aligned}
$$

We can see that $\int_{\frac{i}{n}}^{\frac{i+1}{n}} e(s)\mathrm{d}s$ can be expressed by parameters $\{i, \int_{\frac{i}{n}}^1 \bar{F}_B(b)\mathrm{d}b, \bar{F}_B(s)(s \in (\frac{i}{n}, \frac{i+1}{n}))\}$. The error bound can be calculated during the calculation of $\int f_p^*(s)\mathrm{d}s$. We can adjust the Dynamic Programming algorithm framework to also calculate the error bound.

Use $Sol_i(x,y,z)$ to denote the objective value of the problem:

$$
\begin{aligned}
\max \quad & \int_{\frac{i}{n}}^1 f_p^*(s)\mathrm{d}s + \int_{\frac{i+1}{n}}^1 e(s)\mathrm{d}s \\
\text{s.t.} \quad & x = \bar{F}_B(s) \qquad s \in (\frac{i}{n}, \frac{i+1}{n}) \\
& y = \int_{\frac{i}{n}}^1 \bar{F}_B(b)\mathrm{d}b \\
& z = \int_{\frac{i}{n}}^1 \bar{F}_B(b)^2\mathrm{d}b
\end{aligned}
$$

Sol_i represents the solution set that stores a number of cells, each with indices (x,y,z) and storing the objective value $Sol_i(x,y,z)$.

The adjusted algorithm works as follows. In the first round, calculate $\int_{\frac{n-1}{n}}^{1} f_p^*(s)\mathrm{d}s$. From round 2 to round n, add $\int e(s)ds$ to $\int f_p^*(s)ds$ during the calculation.

In updating the objective from maximizing $(\int_{\frac{i}{n}}^{1} f_p^*(s)\mathrm{d}s + \int_{\frac{i+1}{n}}^{1} e(s)\mathrm{d}s)$ to maximizing $(\int_{\frac{i-1}{n}}^{1} f_p^*(s)\mathrm{d}s + \int_{\frac{i}{n}}^{1} e(s)\mathrm{d}s)$:

Initialize Sol_{i-1}

for each cell in Sol_i :

$$Sol_i^{'}(x, y, z) = Sol_i(x, y, z) + \int_{\frac{i}{n}}^{\frac{i+1}{n}} e(s)ds$$

for $x' \geq x$:

update y', z'

if $Sol_i^{'}(x, y, z) + \int_{\frac{i-1}{n}}^{\frac{i}{n}} f_p^*(s)ds > Sol_{i-1}(x', y', z')$:

$$Sol_{i-1}(x', y', z') = Sol_i^{'}(x, y, z) + \int_{\frac{i-1}{n}}^{\frac{i}{n}} f_p^*(s)ds$$

After round n, for each cell in Sol_0, add $\int_0^{\frac{1}{n}} e(s)ds$ and γ_1. It remains to find the largest value $maxFP\_e$ in Sol_0 and verify that this value ≤ 1. (A limitation of this analysis is around 30% longer running time.)

We measure the new error bound obtained in this way as $maxFP\_e - maxFP$, and compare with the original error bound err. Some experiment results show that this method can reduce the error bound to $60\% - 65\%$ of the original value:

n	$\alpha = 0.7$	$\alpha = 0.71$	$\alpha = 0.72$
35	0.6144	0.6106	0.6067
40	0.6237	0.6206	0.6175
45	0.6304	0.6272	0.6245

Table 3.2: Analysis of error bound: new / original (ratio)

Chapter 4

Work and Limitation: Multi-unit Setting

This project also considers bilateral trade with multiple identical items. [4] proposes the Multi-unit Fixed Price Mechanism. The mechanism picks a fixed unit price p and iteratively offers p to the seller and buyer to trade one more unit until one of the agents rejects. They also show that choosing p from the Generalized Random Quantile Mechanism can achieve a $1 - \frac{1}{e}$ fraction of the optimal welfare, preserving the ratio for the Random Quantile Mechanism in the single-item setting.[1] Now we know mechanisms that give better lower bounds for the one-item case, and we want to know their performance in a generalized multi-unit case.

This project tries to generalize the single-item Fixed-Price Mechanism in [5]. We consider that only G_B is given, and show the existence of a price distribution $\widetilde{f}_p$ with respect to G_B such that for any G_S, the mechanism can guarantee α fraction of the optimal welfare. The unit-price p is sampled from the price distribution $\widetilde{f}_p$.

[1] The single-item Random Quantile Mechanism works as follows: 1. Draw a number x in interval $[\frac{1}{e}, 1]$ from the cumulative distribution function $ln(ex)$ for $x \in [1/e, 1]$. 2. Find $F_S(q(x)) = x$, set the price as $q(x)$. The Generalized Random Quantile Mechanism generalizes this mechanism into the multi-unit setting.

To give an analysis of the lower bound or upper bound of the generalized mechanism requires the characterization of $\widetilde{f}_p$. But currently this project does not find useful characterizations of $\widetilde{f}_p$ for analysis, which is a limitation of this study.

Chapter 5

Conclusion

To conclude, the first part of this project is on Fixed-Price Mechanisms. Following the analysis in [5], we implement the Dynamic Programming algorithm more efficiently in terms of running time and memory use, and make improvement in the analysis of the error bound.

The second part of the work is on generalizing the Fixed-Price Mechanism into the multi-unit setting, but this study does not give bounds on the performance of the generalized mechanism.

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