DETAILED PROJECT PLAN FOR COMP4801 FINAL YEAR PROJECT (UNDERGRADUATE RESEARCH FELLOWSHIP PROGRAMME)

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ABSTRACT. This report is submitted to fulfill the requirement of submitting the detailed project plan in the course COMP4801 **detailed project plan**, as a first deliverable. It provides a thorough explanation of the background, objective, methodology, and schedules & milestones of our project *Computational Examples for Boolean Constraint System Algebras and Nonlocal Games.*

1. Project Background

The main object we are investigating is the Boolean[1] constraint systems (BCS), or a satisfiability problem with finite support of variables taking values in the Boolean domain. General computational problems can be encoded into a system comprised of relations among Boolean variables, or equivalently a BCS. Both computing BCS and deciding the satisfiability of BCS induce different complexity classes, which are building blocks of classical complexity theory [Pap94]. Moreover, each BCS can be associated with a BCS non-local game [PS23], which is a test among two provers and a verifier. The provers are unable to communicate during the game, and they are separately given two questions, to which they shall reply. The players win if the predicate related to the game evaluates to 1 on the 4-tuple of verifier inputs and provers outputs. With shared randomness, the players can play this game perfectly if and only if the underlying BCS is satisfiable, meaning that there is a Boolean-value assignment to its variables satisfying all the constraints. Thus, one can always examine the satisfiability of a BCS by investigating the existence of the perfect strategy of its 'non-local game' version.

However, there are instances of BCS that are not satisfiable in the classical picture, but can be played perfectly by sharing an entangled quantum state between the two provers. The renowned *Mermin-Peres Magic Square Game* is a nice example, first proposed by Mermin and Peres in [Mer90, Per90]. A subsequent work by Cleve and Mittal introduced the general class of BCS games,

[1]We use this word rather than 'binary' to avoid confusion with 2-ary relations.

and gave a condition that ensures the existence of perfect strategy for a special class of these games [CM14].

The satisfiability of a BCS game with a shared quantum state generalizes the classical strategy to a *quantum* one, which is natural since equivalently we are assigning Hermitian and idempotent matrices to variables in the system rather than $\{0,1\}$ values. Cleve and Mittal implicitly included in [CM14] a result that a BCS instance B is satisfiable if and only if its associated $*$-algebra $\mathscr{A}(B)$ has a $*$-homomorphism to the $*$-algebra $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a finite-dimensional Hilbert space. Such an algebra appeared firstly in Ji's work [Ji13], and more recently in Paddock's discussion of approximate rigidity results of BCS non-local games [Pad24].

While non-local games are obtained by encoding a BCS into a multi-interactive proof system, we are particularly interested in a special class of it, *i.e.*, the synchronous games, where the two verifiers share the same input and output spaces, and its predicate requires a 'consistency check'. Central to the research on synchronous games is the associated $*$-algebra $\mathcal{A}(\mathcal{G})$ to any instance $\mathcal{G}$, whose primitive version was the C^*-algebra proposed by Paulsen *et al.*and then formally stated in [HMPS17]. It is shown by Kim, Paulsen and Schafhauser that every BCS non-local game can be transformed into a synchronous game, while preserving the soundness and completeness [KPS18]. By examining the existence of $*$-homomorphism from $\mathcal{A}(\mathcal{G})$ to any $*$-algebra endowed with various properties, one can develop different models of entanglement, *i.e.*, the collection of all the quantum correlations (strategies) among the two provers. To specify, we denote $\mathcal{M}_t$ as the t-models of the framework under which the game is played, and typical choices of t are in $\{loc, q, qa, qc, C^*, hered, alg\}$. By their definitions, there is a hierarchical order among these models, specified by

$$\mathcal{M}_{loc} \subseteq \mathcal{M}_q \subseteq \mathcal{M}_{qa} \subseteq \mathcal{M}_{qc} \subseteq \mathcal{M}_{C^*} \subseteq \mathcal{M}_{hered} \subseteq \mathcal{M}_{alg}.$$

Subsequent works have provided a refinement of the above chain on general non-local games. In [CMN$^+$06] a graph with classical chromatic number 5 but quantum chromatic number 4 is constructed, yielding $\mathcal{M}_{loc} \subsetneq \mathcal{M}_q$. Slofstra [Slo17] proved that $\mathcal{M}_q$ is not closed, resulting the strict inclusion $\mathcal{M}_q \subsetneq \mathcal{M}_{qa}$. Ji, Natarajan, Vidick and Wright proved that $\mathcal{M}_{qa} \neq \mathcal{M}_{qc}$. Paddock and Slofstra constructed a constraint system that is C^*-satisfiable but not tracially-satisfiable in [PS23], or $\mathcal{M}_{qc} \subsetneq \mathcal{M}_{C^*}$. In [CLS17] it is proved that $\mathcal{M}_{qc} = \mathcal{M}_{C^*} = \mathcal{M}_{hered} = \mathcal{M}_{alg}$ for linear systems. In [He24] it is proved that $\mathcal{M}_{C^*} = \mathcal{M}_{hered}$ for any synchronous games, and a computational example for $\mathcal{M}_{hered} \neq \mathcal{M}_{alg}$ is provided.

We focus on the contributions of quantum resources to the classical complexity theory. The two-player(prover) one-round games have been extensively studied in classical complexity theory, and this model induces the complexity class MIP.

With shared quantum states, we have the quantum analog of MIP, *i.e.*, the MIP^*. Researchers have been interested in how powerful these interactive proof systems are, or equivalently, at which hierarchy these communication models fit into among the various complexity classes we currently have. It was first proved by Babai, Fortnow, and Lund [BFL] that $\mathsf{MIP} = \mathsf{NEXP}$. Whether the inclusion $\mathsf{MIP} \subseteq \mathsf{MIP}^*$ is vague for a desirable amount of time, until it was eventually showed true by Ito and Vidick [IV12]. Natarajan and Wright [NW19] showed that the MIP^* system undergoes an exponential growth in power by presenting the result $\mathsf{NEEXP} \subseteq \mathsf{MIP}^*$. The latest result by Ji, Natarajan, Vidick, Wright, and Yuen [JNV+22], which is still under peer review, yields $\mathsf{MIP}^* = \mathsf{RE}$.

There are still many open problems in this field that need to be addressed. and we tend to focus on those being easier to resolve[2]. In [Har23] it is questioned that if two provers can win the synchronous game $\mathcal{G}$ with probability $1-\epsilon$, whether the players can correspondingly win the game $\mathrm{Hom}(G_\lambda, K_3)$ with probability $1-O(\epsilon)$, where G_λ is a asymmetric graph induced from the predicate of $\mathcal{G}$. Moreover, deciding whether $\chi_{alg}(G) = 4$ for any graph G is proved undecidable in the same literature, but a tight classification for this problem into the complexity classes we have insofar investigated is still open. Whether the QMA class, which is an acronym of the Quantum Merlin-Arthur, can fit into $\mathsf{MIP}[q = O(\log n), a = O(\log n)]$ in certain manner, is rendered open in the reviewing preprint [NN24]. Due to the limitation of context, we will not herein give detailed clarification, but left readers to the bibliography where they can find numerous related open problems.

2. Project Objective

We focus on a small fraction of current open problems lying in properties of $*$-algebras of synchronous games and the complexity of variant of MIP systems with quantum resources. The project outcomes would primarily consist of proof of theorems and the development of certain computational tools (`Mathematica`, `Python`, intended). The author, through a private talk with Shunzhuang Huang [Hua], was introduced to the concept of Haar measure that enables the endowment of analytical properties to topological groups. It may potentially provide convenience to examine further characteristics of the $*$-algebra (under Helton *et al.*'s interpretation, is a quotient of a free group algebra [HMPS17]).

3. Project Methodology

There are various toolkits in the field of non-local games. For instance, one can use the NPA-hierarchy method [NPA08] and exhausted search in incremental

[2]There are certain problems that are extremely hard. Concerning the feasibility to solve them as a undergraduate student, we only list relatively easy ones here.

dimensions [JNV+22] to approximate the value of $\omega_{qc}(\mathcal{G})$ and $\omega_q(\mathcal{G})$, respectively. Building upon Watts, Helton and Klep's work [BWHK23], a computational tool to decide the non-existence of perfect C^*-strategy for general non-local game is developed in [He24]. We will moreover attempt to solve open problems by applying mathematical tools introduced in [Arv76, HMPS17, GS24, NW19]. Preliminary knowledges on C^*-algebras, functional analysis, real analysis and basic abstract algebra will be required.

4. Project Schedule and Milestones

Proposing a strict timeline for this project might be rather difficult. We would suggest an obscure one instead.

Table 1. Tentative Schedule for URFP

Date	**Events**
2024 Sep. - Oct.	Literature review, collect attemptable open problems.
2024 Nov. - 2025 Jan.	Learn mathematical tools accordingly.
2025 Jan. - 2025 May.	Start solving the problems.

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